On Concatenations of Algebraic and Spherical Codes on the AWGN Channel

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Abstract

We have investigated a class of concatenations between optimal algebraical and spherical codes, the outer codes being Reed-Solomon codes modified in a special way to lower the complexity of sequential decoding. As an explanation of our empirical results, we discovered a surprising weakness of short non-trivial spherical codes which contrasts to the asymptotical strength of long codes guaranteed by the channel coding theorem. As a result, simple BPSK modulation shows up to be superior to all practically implementable spherical codes for low SNR channels at code rates up to one.

1 Introduction

Reed-Solomon (RS) codes have been shown in [1, 2] to be theoretically optimal as outer codes in concatenated coding systems because they meet the error exponent of a super-channel modelled as Asymptotic Uniform Symmetric Channel (AUSC) within the whole interval of permittable code rates. In typical practical applications, outer RS codes are concatenated with binary inner codes. The inner code bits are then either directly transmitted as BPSK signals or mapped to higher-rate constellations such as QAM using set-partitioning techniques.

We investigated whether it is possible to achieve additional gains on an Additive White Gaussian Noise (AWGN) channel by using more complex signal constellations and directly mapping the symbols of a Galois field $GF(Q)$ to the corresponding signal vectors without using an intermediate binary code. In order to allow for performance comparisons, we restricted our investigations to an overall code rate R of about $1/2$. The inner code rate r was chosen to be at most 1, which seemed to be a reasonable choice because a low outer code rate R would make it impossible to approach the Shannon limit.

All investigated inner codes are spherical, meaning that the energy of all signal vectors is the same—a restriction which does not reduce the minimum euclidean distance $d_{\min}$ too much if the code rate is at most 1. Some of them were taken from [3] where they are proven to maximize $d_{\min}$, but the vast majority were created by numerically maximizing $d_{\min}$ using a Simulated Annealing algorithm.

2 Modifying Reed-Solomon codes for lowering the complexity of sequential decoding

Soft-decision and maximum-likelihood decoding is mandatory for achieving a code performance near the Shannon limit. The Viterbi algorithm, which is very well suited for decoding simple convolutional codes, cannot be used efficiently for decoding complex block codes because its complexity is $O(Q^{N-K})$ where N and K are the code parameters and Q is the cardinality of the underlying Galois field. Since Generalized Minimum Distance (GMD) decoding did not yield any significant improvements over simple hard decision decoding in our special case [4], we finally decided to explore the potential of sequential decoding by implementing the Fano algorithm, the stack algorithm, and the bidirectional stack algorithm [5].

However, Reed-Solomon codes are not very well suited for sequential decoding because their structure is far too symmetric: every combination of K symbols $c_1, \ldots, c_K$ can be extended to a valid code word $\vec{c} = (c_1, \ldots, c_N)$, and thus a sequential decoder cannot take advantage of the algebraical code structure until the last information symbol c_K has been processed. Therefore, it is desirable to insert some additional parity symbols periodically all over the code word in order to help the sequential decoder recognizing potentially wrong code words as early as possible, thereby avoiding unnecessary examination of huge parts of the code tree and reducing decoding complexity. The number of additional parity symbols will be subsequently denoted by L.

Unfortunately, the additional parity symbols will not increase the minimum Hamming distance $D_{H,\min} = N-K+1$ of the code because sequential decoding imposes a strict constraint on the shape of the generator matrix G: Each additional parity symbol c_j must only depend on those information symbols which have been already covered by the preceding code symbols $c_1, \ldots, c_{j-1}$. Encoding the information word $\vec{x} = (0,...,0,1) \in GF(Q)^K$ will thus yield a code word of the form $(0,...,0,1,c_{K+L+1}, \ldots, c_{N+L}) \in GF(Q)^{N+L}$, the Hamming weight of which is $N-K+1$. To put it the other way round: the minimum Hamming distance $D_{H,\min}$ has to be lowered if the code length N and dimension K are to be kept constant. Consequently, it is necessary to find a compromise between minimum distance and decoding complexity. Such a compromise can be obtained from the well-known Gilbert-Varshamov bound that is approximately met by the majority of random linear codes:

$$R = \frac{K}{N}\log_2 Q = \log_2 Q - H_Q\left(\frac{D_{H,\min}}{N}\right) \tag{1}$$

where

$$H_Q(x) = x\log_2(Q-1) - x\log_2 x - (1-x)\log_2(1-x) \tag{2}$$

is the Q-ary entropy function.

3 Empirical results

Substantial performance gains over hard decision decoding were only achieved for relatively small RS codes with $Q = 16$ or $Q = 32$. For $Q = 16$, it was possible to do sequential decoding of unmodified RS codes, yielding a gain of about 2 dB. For longer codes ($Q = 64$), sequential decoding with reasonable complexity required the insertion of so many additional parity symbols that the minimum Hamming distance was lowered below the Gilbert-Varshamov bound, thereby weakening the modified code so much that the impressive sequential decoding gains of up to 4 dB (compared to hard decision decoding of the modified RS code) were almost completely neutralized.

At best, a word error rate (WER) of 10^{-5} could be realized on an AWGN channel with $E_b/N_0 \approx 3.75$ dB, using an inner spherical code with $m = 64$ signal vectors of imension $n = 8$ and an outer modified RS code with $N = 63$, $K = 40$, and $D_{H,\min} = 5$, the overall code length being $\mathcal{N} = Nn = 504$. Using a three-stage concatenation of an outer RS code (hard decision decoding), an intermediate modified RS code (sequential decoding), and an inner spherical code, this result could be improved to about 3.5 dB with $Q = 32$ and $N = 5766$.

These results are far away from the performance of good binary codes such as Turbo-Codes, and they are also far away from the R_0 bound on the AWGN channel ($E_b/N_0 = 1.7$ dB for an overall code rate of $1/2$), which is a theoretical limit for the performance of sequential decoding. Additionally, the ratio of decoder errors to decoder failures (i. e. the ratio of those code words decoded wrong to those which could not be decoded at all) was relatively high due tue the small minimum Hamming distance of the modified codes—a fact that could be problematic in realistic applications. We thus started searching for reasonable explanations of our results.

4 Comparison of superchannels created by nearly optimal spherical codes

4.1 Symbol error rate (SER)

The channel coding theorem states that, for any code rate r below the channel capacity r_c, the symbol error probability (SER) p of an optimal spherical code of length n is asymptotically for $n \to \infty$ given by the expression

$$p = e^{-E(r)n+o(n)} \tag{3}$$

where $E(r)$ is the error exponent of the underlying (AWGN) channel for code rate r, and $o(n)$ is an arbitrary function with $\lim_{n\to\infty}(o(n)/n) = 0$. Hence, the SER tends to zero exponentially for all combinations of E_b/N_0 and code rate below the channel capacity curve.

However, since optimal spherical codes do not have any symmetries which could be exploited to decode them more efficently than with exhaustive search, and since the cardinality m of the code rises exponentially according to the expression $m = 2^{rn}$, it is practically impossible to use (or even to find) such codes for arbitrarily large n. If we restrict m to at most 256, the maximum possible code length is $n = 8$ for code rate $r = 1$, and $n = 16$ for code rate $r = 1/2$.

We examined the SER of short spherical codes with code rates $r = 1/2$, $r = 2/3$, $r = 3/4$, and $r = 1$ by numerical simulation—as an example, the results for $r = 1$ are shown in figure 1. We observed that even for E_b/N_0 values well above the Shannon bound, the SER increases with increasing n, at least for practically feasible values of n, which is contrary to the asymptotical behaviour for large values of n guaranteed by the channel coding theorem. The simple BPSK modulation, which can be considered a trivial spherical code with $m = 2$, $n = 1$, and $r = 1$, yields the lowest SER of all practically usable spherical codes with code rate $r \in [1/2, 1]$ on all AWGN channels with $E_b/N_0 < 3$ dB. A good example for the superiority of BPSK modulation can be given for $r = 1$ and $n = 2$: in this case, the optimal spherical code is equivalent to 4-QAM or 4-PSK modulation, which are well known to be inferior to BPSK.

4.2 Superchannel capacity

Since the cardinality m of the inner code is equal to the cardinality Q of the outer code's symbol alphabet, the increasing SER could be potentially compensated by the better error-correcting capabilites of algebraical codes over large symbol alphabets. Therefore, we

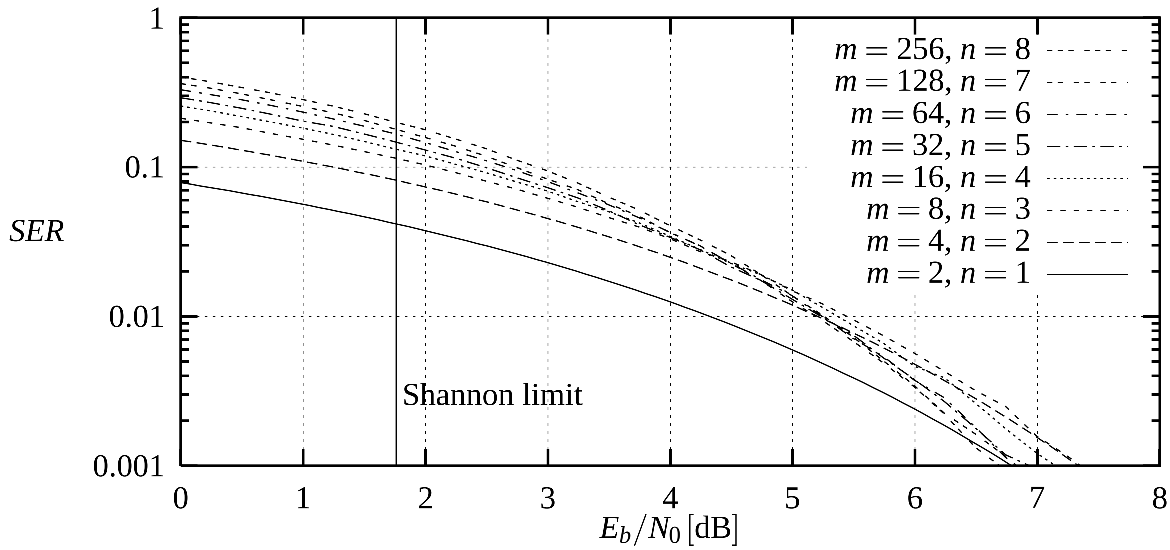

Figure 1: SER of (nearly) optimal short spherical codes of rate $r = 1$.

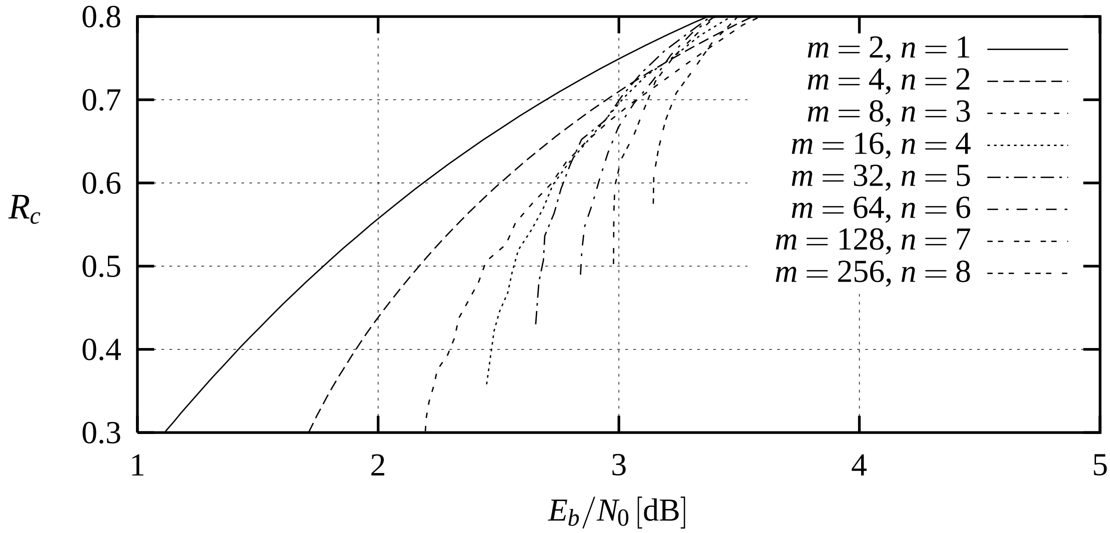

Figure 2: Superchannel capacity of (nearly) optimal short spherical codes of rate $r = 1$.

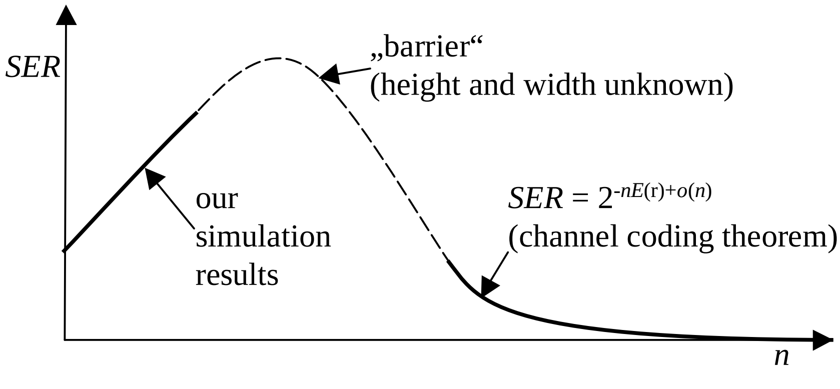

Figure 3: Qualitative dependency between code length n and symbol error rate *SER* of optimal codes on an AWGN channel when the code rate r is close to the channel capacity.

calculated the channel capacity of the superchannels created by the examined spherical codes on an AWGN channel. For sake of simplicity, we modelled each superchannel as Uniform Symmetric Channel (USC), which is the worst-case superchannel for any given SER p.

Each combination of inner code rate r and overall code rate $\mathcal{R}$ corresponds to the outer code rate $R = (K/N)\log_2 Q = (\mathcal{R}/r)\log_2 Q$ where $Q = m$ is the cardinality of the inner code or, equivalently, of the outer code's symbol alphabet. The capacity R_c of a Q-ary USC channel with SER p is given by the equation

$$R_c^{\text{USC}} = \log_2 Q - H_Q(p) \tag{4}$$

which can thus be used to calculate the maximum tolerable SER p for any given combination of r, $\mathcal{R}$, and Q. Finally, this SER can be transformed to the corresponding E_b/N_0 value by using simulation results such as those shown in figure 1.

The resulting channel capacity curves for inner code rate $r = 1$ are shown in figure 2, clearly confirming the superiority of BPSK modulation over a wide range of overall code rates: if, for example, the inner code rate is $r = 1$, and the overall code rate is $\mathcal{R} = 1/2$, then the minimum required E_b/N_0 is about 1.8 dB for BPSK modulation, whereas an inner spherical code with $m = 128$ and $n = 7$ requires an E_b/N_0 of nearly 3 dB.

5 Performance limits for fixed-length concatenations

The preceding results apply to concatenations with arbitrarily long outer codes. We also considered concatenations of fixed overall code length $\mathcal{N}$. In this case, the superiority of BPSK modulation ($n = 1$) is further strenghtened by the fact that it allows longer algebraical codes to be used as outer codes than any spherical code with $n > 1$ because the outer code length N can be computed as $N = \mathcal{N}/n$.

The error exponent $E(R)$ of the superchannel is defined implicitly by $P = e^{-E(R)N+o(N)}$ where P is the word error rate (WER) of the concatenated code. The minimum error exponent required for achieving a given WER P with a concatenated code of overall length $\mathcal{N}$ is accordingly given by

$$E(R) = -\frac{1}{N}\log_2 P = -\frac{n}{\mathcal{N}}\log_2 P. \tag{5}$$

Using the upper and lower bounds from [1] on the error exponent of the USC superchannel, it is now possible to compute upper and lower bounds on the maximum tolerable SER p of the inner spherical code, from which one can finally obtain the minimum required E_b/N_0 of the underlying AWGN channel using our simulation results mentioned in section 4.1.

We computed these E_b/N_0 values for $P = 10^{-5}$, inner code rate $r = 1$, overall code rate $\mathcal{R} = 1/2$, and overall code length $\mathcal{N} = 3700$, which is the minimum length of an optimal code capable of reaching a WER of 10^{-5} on an AWGN channel at $E_b/N_0 = 0.7$ dB [6]: With BPSK modulation, the minimum required E_b/N_0 is 2.7 dB, whereas using an inner spherical code with $m = 256$ and $n = 8$ requires an E_b/N_0 of at least 4.1 dB. Note, however, that these calculations assume hard-decision decoding of the outer code, which explains why some practical coding systems do actually achieve a higher performance using soft-decision decoding techniques.

6 Conclusion

The use of non-binary geometrical codes such as QAM or PSK constellations is mandatory for achieving overall code rates greater than one. Our results show, however, that the use of short non-trivial geometrical codes does not promise to yield improvements over simple BPSK modulation if the overall code rate is to be less than one. This is due to the fact that, if the code rate r is sufficiently close to the Shannon bound, the error probability characteristics of short spherical codes are different from the asymptotical behaviour of long codes given by the channel coding theorem: In order to obtain the SER reduction guaranteed by the channel coding theorem for $n \to \infty$, it is first necessary to overcome a high SER "barrier" at low values of n, as shown in figure 3, which is impossible if n is restricted to practically feasible values. From this point of view, it is not surprising that the best known channel coding systems with low code rates are based on binary codes and BPSK modulation.

There are, however, some questions remaining: So far, we do not have any theoretical explanation for the existence of the SER "barrier" mentioned above, nor do we have any idea of how high and wide this barrier actually is, i. e., how large the code length n must become until the SER has reached its maximum value and finally starts decreasing exponentially according to the channel coding theorem. Finding an answer to these questions raised by the results of our empirical work could be an interesting topic for more theoretically oriented research in the future.

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